Differential Equation for the Amplitude of Wrinkles

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Introduction

THE renewed interest in the phenomenon of wrinkling of membranes, fueled by applications to the aerospace industry, has spurred considerable activity on both the theoretical and computational aspects of the subject. Although the wrinkling phenomenon is known to be a case of shell buckling, it seems unwise to utilize this knowledge in actual computations because treating the membrane as a shell with a vanishingly small thickness leads to the frustrating effects of numerical instability. There exist, on the other hand, techniques based on the membrane theory alone that permit us to obtain, with relative ease and numerical robustness, not only the extent of the wrinkled zones, but also the field of directions of the wrinkles.[1–4] All that is needed, therefore, is a means for calculating the amplitude and wavelength of the wrinkles in an already known wrinkled domain. In this Note, we derive an ordinary differential equation whose solution delivers such information. For clarity of the presentation, the equation is obtained under a number of simplifying assumptions, mostly involving the smallness of the deformations. However, the method is straightforwardly extendable to the general case.

Our guiding principle has been to keep only those aspects of the model that are clearly determinant of the result, while leaving all other aspects for future refinement. We believe that this philosophy will serve to highlight the essential features, rather than the technicalities, of the proposed equation. Note that, even under a geometrically and materially linear regime, the resulting differential equation is highly nonlinear. The physical reason for this nonlinearity is to be found in that, because wrinkling entails no stress in the direction perpendicular to the wrinkles, there exists a severe constraint between the wavelength and amplitude of the wrinkles. This can be phrased intuitively as follows: For a given amount of wrinkling strain, the larger the number of wrinkles, the smaller their amplitude. To the best of our knowledge, the only attempt in a direction similar to the one proposed herein is due to Wong and Pellegrino.[5] In their work, these authors present a model based mainly on the particular case of shearing of a rectangular panel. Their results are very valuable and will be used here as a benchmark for validation purposes.

Derivation of the Equation

Assume that a given domain $\mathcal{D}$ of a membrane is known to be unidirectionally wrinkled, and let x denote the known direction of the wrinkles within the domain. For the sake of capturing the essence of the phenomenon, we will neglect the effect of the overall curvature of the membrane within this domain. Moreover, we will assume that, in the domain of interest, the wrinkles run parallel to each other, with y denoting the direction orthogonal to the wrinkles. As pointed out in the Introduction, these restrictions are easily removed, but their inclusion complicates the formulas without necessarily contributing significantly to the understanding of the phenomenon at hand.

Let the wrinkling strain field be denoted by $\epsilon_w = \epsilon_w(x, y)$. This field is assumed to be known from a "standard" membrane analysis. The magnitude of ϵ_w is assumed to be small, thus avoiding the use of the exact Lagrangian strain, again for simplicity. Let λ denote the wavelength of the wrinkles. We conceive of λ as a smooth field $\lambda = \lambda(x, y)$ over the wrinkled domain $\mathcal{D}$. The wrinkles themselves are described by the formula

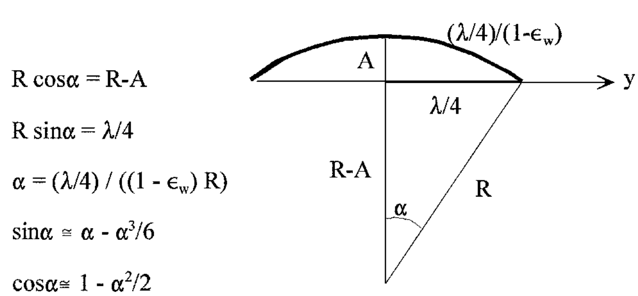

Fig. 1 Relationship between wrinkling strain, curvature, amplitude, and wavelength (x axis is perpendicular to the plane of Fig. 1).

$$w = A \sin(2\pi y/\lambda) \tag{1}$$

where the amplitude A is also to be regarded as a smooth field $A = A(x, y)$. An accurate analysis of the length of the sinusoidal curve can be carried out in terms of elliptic integrals, but to be consistent with our desire to simplify matters without sacrificing the essential elements, we will assimilate each half-wave to a circular arc. Assuming that the wrinkling leaves the material lengths in the y direction unchanged, we can easily obtain (Fig. 1) the following approximate relations between the amplitude A, the (mean) radius of curvature R (with $R \gg A$), the wavelength λ, and the wrinkling strain ϵ_w (assumed positive in contraction):

$$AR = \lambda^2/32, \qquad 6\epsilon_w R^2 = \lambda^2/16 \tag{2}$$

where only the first significant terms of the relevant Taylor expansions have been retained. Combining the preceding expressions and denoting the curvature by $\kappa = 1/R$, we obtain the relation

$$A\kappa = 3\epsilon_w \tag{3}$$

A more accurate analysis based on elliptic integrals affects only the numerical coefficient in the right-hand side (from 3 to approximately 4.63), but not the essence of the conclusion, namely: For a given wrinkling strain, the product of the amplitude times the curvature is constant.

The elastic energy stored in the wrinkles will be attributed to a double provenance. The first is obvious, namely, the curvature κ just introduced. This energy can be evaluated as

$$\mathcal{W}_1 = \iint_{\mathcal{D}} \frac{1}{2} D\kappa^2 \, dx \, dy = \iint_{\mathcal{D}} \frac{1}{2} D\left(\frac{3\epsilon_w}{A}\right)^2 dx \, dy \tag{4}$$

where D is the bending stiffness of the membrane (viewed as a shell). Had we not assumed that the wrinkles are parallel, the Jacobian determinant of the correspondingly skewed coordinate system would have had to have been introduced. Moreover, if the true sinusoidal wave shape had been used by means of an elliptic integral, then an averaging of the curvature would have led to a factor of approximately 2.31 instead of 3 in the preceding expression.

The second source of elastic energy storage is that, should there be any x slope of, for example, the crest of a wrinkle, the longitudinal tensile strain ϵ_0 would increase by the small amount $\frac{1}{2}(A'^2)$, where a prime is used to denote partial x derivatives. If $A'^2 \ll \epsilon_0$, this effect results in an addition to the energy in the amount

$$\mathcal{W}_2 = \iint_{\mathcal{D}} \frac{1}{2} F\epsilon_0 w'^2 \, dx \, dy = \iint_{\mathcal{D}} \frac{1}{4} F\epsilon_0 A'^2 \, dx \, dy \tag{5}$$

where F represents the in-plane stiffness of the membrane. The passage from w to A, with the consequent change in the numerical coefficient, has the following explanation. If one uses the amplitude of the wave to gauge the slope (at the crest), then a mean-square

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should be used to take into consideration the sinusoidal variation, just as in calculating the power of an alternating current.

Note that a third source of energy could be found in the second derivative A'', which is responsible for the appearance of a curvature in the x direction. This source can be included in the theory, particularly to take care of the possibly large curvatures that may arise in the boundary layer between the wrinkled and the unwrinkled domains. It will be neglected in the present analysis, but its effect is more than merely cosmetic: The differential equation would be of the fourth, rather than the second, order.

The membrane will naturally select a wrinkle pattern that minimizes the sum of the extra energy $\mathcal{W}_1 + \mathcal{W}_2$, thus reducing the problem to one in the calculus of variations. Taking the variations and integrating by parts in the standard way yields the differential equation

$$A^3 A'' = -\frac{18\epsilon_w^2 D}{F\epsilon_0} \tag{6}$$

Note that this is a partial differential equation, but it is reduced to an ordinary equation to be solved on each line $y = \text{constant}$. Appropriate boundary conditions are usually the vanishing of A at the boundary of the wrinkled domain. Denoting by t the thickness of the membrane and using the standard linear elastic constants $D = Et^3/[12(1-\nu^2)]$ and $F = Et$ (E = Young's modulus, ν = Poisson's ratio), we can write

$$A^3 A'' = -\frac{3\epsilon_w^2 t^2}{2\epsilon_0(1-\nu^2)} \tag{7}$$

We have obtained a nonlinear ordinary differential equation governing the amplitude of the wrinkles. A more careful analysis will result in some changes to the numerical value of the constants appearing in the right-hand side and/or in additional terms, but we have captured the essential nonlinearity of the phenomenon under the umbrella of a relatively simple-looking equation. If, for instance, the assumption that A'^2 is very small when compared with the longitudinal stretch ϵ_0 is dropped, the resulting differential equation becomes

$$(\epsilon_0 - 3A'^2)A^3 A'' = -\frac{3\epsilon_w^2 t^2}{2(1-\nu^2)} \tag{8}$$

Analysis and Validation

The fundamental equation (7) is nonlinear, and there exists no general expression for its solution, which may not even exist under certain circumstances. In other words, for a given wrinkling strain field and longitudinal stretch, the task of finding a solution is to be tackled numerically. A particular case of interest, and one that serves to shed light on the general behavior of the solutions, is that for which the right-hand side of the equation is constant. For this case, fortunately, the solution can be found explicitly in analytic form. It is worthwhile anticipating that the vanishing of the amplitude A at the boundaries may require that, for the product A^3A'' to be finite, the slope at both ends of the wrinkle must be infinite. This verticality of the slope is not a defect of the equation itself, but only a consequence of the wrinkling strain field having been assumed constant. In reality, this strain vanishes at the boundary, thus affecting the shape of the wrinkle in the boundary layer. If the given wrinkling strain, though vanishing at the boundary, is substantially constant in the interior of the wrinkled domain, then it is to be expected that the analytical solution that follows these remarks constitutes a faithful representation of the wrinkle wavelength and amplitude away from the boundary layer. Having made these preliminary comments, we seek an analytic solution for the nonlinear ordinary differential equation:

$$A^3 A'' = -k^2 \tag{9}$$

where, in our case, the constant k is given by

$$k = \sqrt{\frac{3\epsilon_w^2 t^2}{2\epsilon_0(1-\nu^2)}} \tag{10}$$

The boundary conditions are $A(0) = A(L) = 0$, where L is the known total length of the wrinkle. The surprisingly simple solution of this problem is given by

$$A = \sqrt{2 * k * L * (\xi - \xi^2)} \tag{11}$$

as is easily checked by direct substitution. We have introduced the nondimensional length variable $\xi = x/L$. In fact, the graph of this solution is an ellipse with semi-axes equal to $L/2$ and $\sqrt{(k * L/2)}$. (Note that a lengthwise half-sine wrinkle shape would correspond to a wrinkling strain that builds up slowly following a squared sine law). We have now a definite expression that can be used to compare this prediction with the results of Wong and Pellegrino.[5] Consider a square membrane of side H whose perimeter is attached to a rigid frame pin jointed at the corners. A shear strain is imposed on the membrane by deforming the boundary frame by an angular amount γ into a parallelogram. Assuming that this experiment imposes on the membrane a constant wrinkling strain (this is only approximately true), we can apply the result embodied in our explicit solution (11). If the membrane is made of a metal and no initial prestress is imposed, then it is clear that the longitudinal strain ϵ_0 (in a diagonal direction) is, at all instants, equal numerically to one-half the amount of shear ($\epsilon_0 = \gamma/2$, for small strains). As far as the wrinkling strain is concerned, we observe that the total contractile strain of the diagonal that shortens is also equal to $\gamma/2$, but it is composed of the wrinkling strain plus the Poisson effect due to the presence of the longitudinal stress. Thus, we may write $\epsilon_w = 0.5\gamma(1-\nu)$. By the use of this value in Eq. (10), the amplitude of the wrinkles at the middle point of the plate ($\xi = 0.5$) is obtained from Eq. (11) as

$$A = \sqrt{\frac{k\sqrt{2}H}{2}} = \sqrt{\sqrt{\frac{3\gamma}{4(1-\nu^2)}}(1-\nu)\frac{\sqrt{2}Ht}{2}} \tag{12}$$

The result reported by Wong and Pellegrino[5] can be expressed as

$$A = \sqrt{\sqrt{\frac{\gamma}{3(1-\nu^2)}}\frac{2(1-\nu)Ht}{\pi}} \tag{13}$$

These results (as well as the corresponding results for the wavelengths) are identical in the sense that they both show a dependence of the amplitude on the square root of the length and the thickness, as well as on the quartic root of the imposed shearing strain. The dependence on Poisson's ratio is also identical. The only difference lies in the numerical coefficient. The published coefficient[5] is about 80% of the one obtained by means of the present analysis. Given the various simplifying assumptions used in both approaches, this is quite a remarkable fit because what matters is the identical functional dependence on the variables and that both results are of the same order of magnitude. On the other hand, the results presented here are derived as a particular case of a general equation that can be applied to any given wrinkling regime.

Conclusions

Now that a differential equation that serves to determine the number and amplitude of elastic wrinkles has been obtained, and given the verification that its solutions match previously available results, the task ahead is double. First, it is imperative to extend the derivation of the equation to the geometrically and materially nonlinear regime. Second, once this task has been carried out, one should incorporate the evaluation of the wrinkle amplitude into a computational tool that performs the standard analysis to determine the wrinkling domain and the direction of the wrinkles. The two computations can be affected in tandem, as suggested in this Note, or possibly integrated into one and the same iterative algorithm.

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Contact Stress from Asymptotic Reissner–Mindlin Plate Theory

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Introduction

PUBLISHED work that goes back decades shows conclusively that Euler–Bernoulli beam theory and Kirchhoff plate theory both fail, even qualitatively, to capture the behavior of transverse shear-stress resultants and contact stresses, such as when a beam or plate is being pressed against a flat surface. When the effects of transverse shear stresses are added to the theories, resulting in Timoshenko beam theory and Reissner–Mindlin plate theory, the accuracy of contact stresses and transverse shear stress resultants is improved.[1–3] In particular, the qualitatively incorrect result of discontinuous transverse shear-stress resultants is overcome, but the contact stresses remain discontinuous. However, in more recent work models that include degrees of freedom associated with transverse normal strain have been shown to yield continuous contact stresses.[4,5] Unfortunately, however, such models are considerably more complex than the classical theories just mentioned (for example, see Refs. 6 and 7).

Recently, beam[8] and plate[9] theories have been derived from three-dimensional elasticity using the variational-asymptotic method. In the plate theories the small parameter h/ℓ is used to reduce the dimensionality of the model, where h is the thickness of the plate and ℓ is the wavelength of deformation in the plane of the plate. A model that is derived in this manner and is valid to order $(h/\ell)^0$ has the form of a Kirchhoff or classical laminated plate theory, but it is not subject to the usual (and internally inconsistent) assumptions that the transverse normal stress is zero and that the normal line element remains straight, of constant length, and normal to the deformed plate. Instead, the normal line element deforms, even in isotropic plates, as a result of the Poisson effect. When all terms through order $(h/\ell)^2$ are kept, the resulting theory, whether for homogeneous, isotropic or laminated, composite plates, can be uniquely cast into the form of a Reissner–Mindlin plate theory. Additional warping of the normal line element occurs as a result of transverse shear effects. When all terms through order $(h/\ell)^2$ are kept in the asymptotic approximations for the three-dimensional field variables, accurate through-the-thickness distributions of all in-plane, transverse shear, and transverse normal strain and stress are obtained. These results are especially helpful in recovery of stress and strain components for laminated plates[10] or shells[11] and can be obtained without introducing any degrees of freedom beyond those of Reissner–Mindlin theory.

Thus, an equivalent single-layer theory for laminated composite plates, when consistently derived using the variational-asymptotic method, is capable of far more than just accurate prediction of global behavior. There is one additional feature of this type of theory that has been developed for plates, although not yet for beams. When the plate is loaded with tractions on its upper and lower surfaces and body forces, the potential of the applied loads will contain terms that arise because of the warping of the normal line element. Indeed, the purpose of this Note is to show that an asymptotically correct theory through order $(h/\ell)^2$, which includes such terms, is sufficient to obtain qualitatively correct transverse shear-stress resultants and transverse normal contact stresses. Contrary to Refs. 4 and 5, additional degrees of freedom to take into account transverse normal strain are unnecessary. The implication of this statement is simply that a theory with no more degrees of freedom than are found in so-called first-order shear deformation theory is sufficient if it includes the contributions of the warping displacement to the potential of the applied loads, which are of the same order as terms in the strain energy caused by transverse shear deformation. This is well within the realm of the more classical theories and leads to the expectation that simpler and hence more computationally efficient theories can be used in problems involving contact.

Although recovery relations are published for nonlinear analysis of laminated plates and shells, to present an analytical solution herein the development is limited to the linear theory of isotropic plates. The total potential for isotropic plates is presented in the next section. Results obtained for a loaded plate undergoing cylindrical bending and being pressed against a rigid, smooth surface are used to illustrate the concept.

Total Potential

For small displacements the total potential for a laminated, composite plate can be written as[9]

$$U + V = \tfrac{1}{2}\mathcal{R}^T A\mathcal{R} + \tfrac{1}{2}\gamma^T G\gamma + \mathcal{R}^T F - v^T f - \phi^T m \quad (1)$$

where U is the strain energy per unit area; V is the potential of applied loads per unit area; $\mathcal{R}$ is a column matrix containing the three membrane measures (ϵ_{11}, $2\epsilon_{12}$, and ϵ_{22}) and the three bending-twist measures (κ_{11}, $2\kappa_{12}$, κ_{22}) of plate deformation; γ is a column matrix containing the two plate transverse shear measures, $2\gamma_{13}$ and $2\gamma_{23}$; v is a column matrix that contains the three displacement components of the plate averaged through the thickness; and ϕ is a column matrix that contains two rotation measures associated with the normal line element. The matrices A, G, and F are thus functions of the number of layers and the material properties of each layer; f, m, and F are also functions of upper and lower surface tractions and body forces. The matrix A is the well-known 6×6 matrix of lamination theory, whereas the others are determined by the variational-asymptotic method. Details are found in Ref. 9.

For the purposes of the present Note, we specialize the theory to the case of isotropic materials and cylindrical bending so that

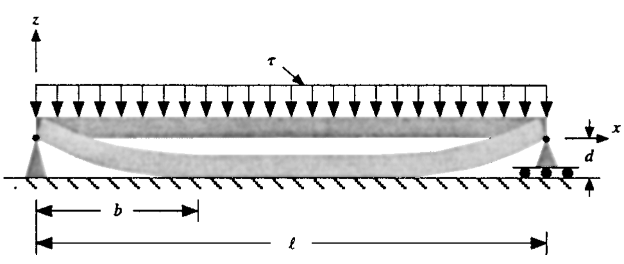

Fig. 1 Schematic of cylindrical bending of a plate, looking along the infinite direction y, contacting a rigid, smooth surface between $b \leq x \leq \ell - b$; τ is shown as negative, and the contact force per unit area β, acting $b \leq x \leq \ell - b$, is not shown.

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